

Quantum gravitational correction to the Hawking temperature from the Lemaître–Tolman–Bondi model

Rabin Banerjee,^{1,*} Claus Kiefer,^{2,†} and Bibhas Ranjan Majhi^{1,‡}

¹*S. N. Bose National Centre for Basic Sciences,
JD Block, Sector III, Salt Lake, Kolkata-700098, India*

²*Institut für Theoretische Physik, Universität zu Köln, Zùlpicher Strasse 77, 50937 Köln, Germany*
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We solve the quantum constraint equations of the Lemaître–Tolman–Bondi model in a semiclassical approximation in which an expansion is performed with respect to the Planck length. We recover in this way the standard expression for the Hawking temperature as well as its first quantum gravitational correction. We then interpret this correction in terms of the one-loop trace anomaly of the energy–momentum tensor and thereby make contact with earlier work on quantum black holes.

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In the absence of a full quantum theory of gravity, it is of interest to consider models which could serve as a possible guide in the construction of such a theory [1]. One such model is the Lemaître–Tolman–Bondi (LTB) model describing the dynamics of a spherically-symmetric dust cloud [2]. It has been already used in a variety of papers dealing with canonical quantization in both the Wheeler–DeWitt framework and loop quantum cosmology, cf. [3–7] and the references therein. While the full quantization of the LTB model has not yet been achieved, it was at least possible to get insights into the recovery of Hawking radiation and black-hole entropy from it.

Our present paper is a continuation of this earlier work. Our motivation is twofold. First, we want to derive a quantum gravitational correction to the Hawking temperature through a semiclassical expansion scheme for the quantum states. Second, we want to present an interpretation of these correction terms through the “trace anomaly” of the matter energy–momentum tensor, making thereby a connection to earlier work [8–13] in the context of black holes.

Let us first present the LTB model. The spherical gravitational collapse of a dust cloud, in an asymptotically flat space-time, having energy density $\epsilon(\tau, \rho)$, is described in comoving coordinates $(\tau, \rho, \theta, \phi)$ by the LTB metric,

$$ds^2 = -d\tau^2 + \frac{(\partial_\rho R(\rho, \tau))^2}{1 + 2E(\rho)} d\rho^2 + R^2(\rho, \tau) d\Omega^2. \quad (1)$$

Inserting this expression into the Einstein field equations leads to, for vanishing cosmological constant,

$$8\pi G\epsilon(\tau, \rho) = \frac{\partial_\rho F}{R^2 \partial_\rho R} \quad (2)$$

and

$$(\partial_\tau R)^2 = \frac{F}{R} + 2E \equiv 1 - \mathcal{F} + 2E, \quad (3)$$

where $F(\rho) \equiv 2GM(\rho)$, with $M(\rho)$ being the active gravitational mass within a $\rho = \text{constant}$ shell, and

$$\mathcal{F} \equiv 1 - \frac{F}{R}. \quad (4)$$

The function $E(\rho)$ is the total energy per unit mass within the same shell; the marginally bound models are defined by $E(\rho) \equiv 0$. The case of collapse is described by $\partial_\tau R(\tau, \rho) < 0$. We set $c = 1$ throughout.

The canonical quantization of the LTB model was developed in [3] and then applied to quantization in a series of papers, see [4, 5, 7]. Although no full quantization has yet been performed, interesting results have been obtained at the semiclassical level; they include the recovery of Hawking radiation plus greybody corrections from solutions to the Wheeler–DeWitt equation and the momentum constraints (that is, the quantum constraint equations). Insights into the microscopic interpretation of black-hole entropy were also obtained, cf. [14] and the references therein.

The semiclassical approximation scheme is also employed here. We start with the quantum constraint equations [4],

$$l_P^4 \frac{\delta^2 \Psi}{\delta \tau^2} + l_P^4 \mathcal{F} \frac{\delta^2 \Psi}{\delta R^2} + \frac{\Gamma^2}{4\mathcal{F}} \Psi = 0, \quad (5)$$

$$\tau' \frac{\delta \Psi}{\delta \tau} + R' \frac{\delta \Psi}{\delta R} - \Gamma \left(\frac{\delta \Psi}{\delta \Gamma} \right)' = 0, \quad (6)$$

where Ψ is a functional of the dust variable $\tau(r)$ as well as the gravitational variables $R(r)$ and $\Gamma(r)$, and $l_P = \sqrt{G\hbar}$ is the Planck length. Here, r is the radial variable in the ADM formalism [3]; we recall that $\Gamma \equiv F' \equiv 2GM'$. We note that (5) is elliptic outside the horizon and hyperbolic inside the horizon; this can be recognized from (4). In contrast to [4, 5] and other papers, no additional factor ordering terms are taken into account here; this will be crucial in obtaining our results.

*Electronic address: rabin@bose.res.in

†Electronic address: kiefer@thp.uni-koeln.de

‡Electronic address: bibhas@bose.res.in

We now make the ansatz

$$\Psi[\tau(r), R(r), \Gamma(r)] = \exp\left(\frac{i}{2l_P^2} \int dr \Gamma S(R, \tau)\right), \quad (7)$$

where $S(R, \tau)$ is a function to be determined recursively in the following semiclassical approximation scheme. With the special factor ordering chosen in [4, 5], this ansatz would lead to an *exact* solution of (5) and (6) with semiclassical form. Here, instead, we shall use this ansatz to solve (5) in a semiclassical approximation; the diffeomorphism constraint (6) is already solved identically with this ansatz.

Inserting (7) into (5), we arrive at

$$\begin{aligned} & \frac{\Gamma}{2} \left(\frac{\partial S}{\partial \tau} \right)^2 + \frac{\Gamma}{2} \mathcal{F} \left(\frac{\partial S}{\partial R} \right)^2 - \frac{\Gamma}{2\mathcal{F}} \\ & - i l_P^2 \delta(0) \frac{\partial^2 S}{\partial \tau^2} - i l_P^2 \delta(0) \mathcal{F} \frac{\partial^2 S}{\partial R^2} = 0. \end{aligned} \quad (8)$$

Here, “ $\delta(0)$ ” indicates the presence of undefined expressions arising from $\lim_{\bar{r} \rightarrow r} \delta(r - \bar{r})$, which require the presence of a regularization scheme. (With the special factor ordering chosen in [4, 5], the $\delta(0)$ -terms are automatically cancelled; this is why they were introduced there.)

For the semiclassical approximation scheme, we make the ansatz

$$S = S_0 + l_P^2 S_1 + l_P^4 S_2 + \dots \quad (9)$$

and compare consecutive orders in l_P^2 . (The general scheme for such an approximation in quantum geometrodynamics is presented in [1, 15].) Since Γ is dimensionless, we see from (7) that S has the dimension of a length (L), so that the dimension of S_1 is L^{-1} , the dimension of S_2 is L^{-3} , and so on.

Inserting (9) into (8) and comparing different orders in l_P^2 , we obtain

$$\mathcal{O}(l_P^0): \quad \left(\frac{\partial S_0}{\partial \tau} \right)^2 + \mathcal{F} \left(\frac{\partial S_0}{\partial R} \right)^2 - \mathcal{F}^{-1} = 0, \quad (10)$$

which, not surprisingly, is equivalent to the Hamilton–Jacobi equation for the action

$$\mathcal{A} = \frac{1}{2G} \int dr \Gamma S(R, \tau). \quad (11)$$

The next order yields

$$\begin{aligned} \mathcal{O}(l_P^2): \quad & \Gamma \frac{\partial S_0}{\partial \tau} \frac{\partial S_1}{\partial \tau} + \Gamma \mathcal{F} \frac{\partial S_0}{\partial R} \frac{\partial S_1}{\partial R} \\ & - i \delta(0) \frac{\partial^2 S_0}{\partial \tau^2} - i \delta(0) \mathcal{F} \frac{\partial^2 S_0}{\partial R^2} = 0. \end{aligned} \quad (12)$$

The following order then involves S_2 ,

$$\begin{aligned} \mathcal{O}(l_P^4): \quad & \frac{\Gamma}{2} \left[\left(\frac{\partial S_1}{\partial \tau} \right)^2 + 2 \frac{\partial S_0}{\partial \tau} \frac{\partial S_2}{\partial \tau} \right] \\ & + \frac{\Gamma \mathcal{F}}{2} \left[\left(\frac{\partial S_1}{\partial R} \right)^2 + 2 \frac{\partial S_0}{\partial R} \frac{\partial S_2}{\partial R} \right] \\ & - i \delta(0) \frac{\partial^2 S_1}{\partial \tau^2} - i \delta(0) \mathcal{F} \frac{\partial^2 S_1}{\partial R^2} = 0. \end{aligned} \quad (13)$$

The solution to the Hamilton–Jacobi equation (10) is

$$S_0 = \pm \left(a\tau + \int^R dR \frac{\sqrt{1 - a^2 \mathcal{F}}}{\mathcal{F}} \right) + \text{constant}, \quad (14)$$

where $a = 1/\sqrt{1 + 2E}$. With the special factor ordering chosen in [4, 5], this would already be the exact solution to the Wheeler–DeWitt equation (this, again, was part of the motivation to choose that factor ordering in the first place). Here, however, the solution (14) occurs at the highest order and must be used in the next-order equations.

The variable a above is seen to be related to the (dimensionless) energy E . This is similar to what happens for the tunneling mechanism [10, 16]. The corresponding ansatz there involves ωt in place of $a\tau$, where t is the Schwarzschild time and ω is identified with the conserved quantity (in this case, the energy) corresponding to the time-like Killing vector. We also note from (14) since a is dimensionless and S_0 has dimension of L , τ has also the dimension of L . This will be used later on.

Observe that the solution (14) also holds in the presence of a cosmological constant Λ , with \mathcal{F} then given by $\mathcal{F} = 1 - F/R - \Lambda R^2/3$ instead of (4), see [7] for the treatment of positive Λ and [6] for the treatment of negative Λ . In that case, however, the integral can only be evaluated in terms of elliptic functions. Furthermore, the constant in (14) only enters the unknown total normalization for Ψ and will therefore be skipped in what follows.

Inserting now (14) into (12), we arrive at an equation for S_1 ,

$$a\Gamma \frac{\partial S_1}{\partial \tau} + \Gamma \sqrt{1 - a^2 \mathcal{F}} \frac{\partial S_1}{\partial R} + i \delta(0) \frac{F}{R^2} \frac{2 - a^2 \mathcal{F}}{2\mathcal{F} \sqrt{1 - a^2 \mathcal{F}}} = 0. \quad (15)$$

We solve this equation by the special ansatz

$$S_1 = -C\tau - iU_1(R), \quad (16)$$

where C is a new variable with the dimension L^{-2} because S_1 has dimension L^{-1} and τ has dimension L ; it will play a crucial role below. Since the only length scale in this model is GM , we can set for later purpose

$$C \equiv \frac{\alpha_1}{(GM)^2}, \quad (17)$$

where α_1 is a dimensionless constant.

Inserting (16) into (15), we get a differential equation for $U_1(R)$,

$$\frac{dU_1}{dR} = \frac{iaC}{\sqrt{1 - a^2 \mathcal{F}}} + \delta(0) \frac{F}{\Gamma R^2} \frac{2 - a^2 \mathcal{F}}{2\mathcal{F}(1 - a^2 \mathcal{F})}. \quad (18)$$

We remark that in the marginal limit, $a \rightarrow 1$, this equation reads

$$\frac{dU_1}{dR} = iC \sqrt{\frac{R}{F}} + \frac{\delta(0)}{2\Gamma(R - F)} \left(1 + \frac{F}{R} \right).$$

Integrating (18), we find apart from an irrelevant constant the desired expression for S_1 ,

$$S_1(R, \tau) = -C\tau + aC \left[\frac{\sqrt{R([1-a^2]R + a^2F)}}{1-a^2} - \frac{a^2F}{(1-a^2)^{3/2}} \ln \left(\sqrt{(1-a^2)R} + \sqrt{(1-a^2)R + a^2F} \right) \right] - i \frac{\delta(0)}{2\Gamma} (2 \ln(R-F) - \ln(R + a^2[F-R]) - \ln R) \quad (19)$$

We also give the special result for the marginal limit, $a \rightarrow 1$:

$$S_1(R, \tau) = -C\tau + \frac{2CR^{3/2}}{3\sqrt{F}} - i \frac{\delta(0)}{2\Gamma} (2 \ln(R-F) - \ln R) . \quad (20)$$

In the next order, $\mathcal{O}(l_P^4)$, we have to insert the solution (19), apart from (14), into (13). This yields a rather complicated equation for S_2 . For example, in the special case $a = 1$, it is of the form

$$\begin{aligned} & \frac{\Gamma}{2} \left(C^2 \pm 2 \frac{\partial S_2}{\partial \tau} \right) \\ & + \frac{\Gamma \mathcal{F}}{2} \left(\left[\frac{C}{\sqrt{1-\mathcal{F}}} - \frac{i\delta(0)}{2\Gamma} \frac{R+F}{R(R-F)} \right]^2 \pm 2 \frac{\sqrt{R\mathcal{F}}}{R-F} \frac{\partial S_2}{\partial R} \right) \\ & - i\delta(0)\mathcal{F} \left(\frac{CF}{2R^2} \left[\frac{F}{R} \right]^{-3/2} - \frac{i\delta(0)}{2\Gamma} \frac{(R-F)^2 - 2R^2}{R^2(R-F)^2} \right) \\ & = 0 . \end{aligned} \quad (21)$$

In analogy to (16), one could try to solve it with an ansatz of the form

$$S_2(R, \tau) = -D\tau - iU_2(R) , \quad (22)$$

where $D \equiv \alpha_2/(GM)^4$ has dimension L^{-4} and involves another dimensionless constant α_2 . We shall not, however, follow this here and restrict our attention only to the order l_P^2 .

Collecting the solutions up to $\mathcal{O}(l_P^2)$, we can write

$$S = S_0 + l_P^2 S_1 = (\pm a - l_P^2 C)\tau \pm \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} + l_P^2 a C \tilde{G}(R) - i l_P^2 \delta(0) \tilde{H}(R) , \quad (23)$$

where

$$\begin{aligned} \tilde{G}(R) & \equiv \frac{\sqrt{R([1-a^2]R + a^2F)}}{1-a^2} - \\ & \frac{a^2F}{(1-a^2)^{3/2}} \ln \left(\sqrt{(1-a^2)R} + \sqrt{(1-a^2)R + a^2F} \right) , \end{aligned} \quad (24)$$

and

$$\tilde{H}(R) = \frac{1}{2\Gamma} (2 \ln(R-F) - \ln(R + a^2[F-R]) - \ln R) . \quad (25)$$

In analogy to earlier papers, cf. [7], we define positive and negative energy states according to the sign in front of the dust proper time variable τ , with the case of the minus sign corresponding to positive energy. Inserting (23) into the general ansatz (7), the positive-energy solution reads

$$\Psi^+ = \exp \left(\frac{i}{2l_P^2} \int d\tau \Gamma \left[-a\tau - \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} - l_P^2 C\tau + l_P^2 a C \tilde{G} - i l_P^2 \delta(0) \tilde{H} \right] \right) , \quad (26)$$

while the negative-energy solution is given by

$$\Psi^- = \exp \left(\frac{i}{2l_P^2} \int d\tau \Gamma \left[a\tau + \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} - l_P^2 C\tau + l_P^2 a C \tilde{G} - i l_P^2 \delta(0) \tilde{H} \right] \right) . \quad (27)$$

In order to calculate the Hawking radiation, we shall evaluate the overlap between the “outgoing dust state with negative energy” Ψ_e^- (where the index “e” refers to “expanding” cloud) and the “ingoing dust state with positive energy” Ψ_c^+ (where the index c refers to “collapsing” cloud). Since the interpretation of these states is made with respect to an observer in the asymptotic regime using the Killing time T , we have to substitute the dust time τ by T [7]. For the outgoing case, we have the relation

$$T = a\tau + \int dR \frac{\sqrt{1-\mathcal{F}a^2}}{\mathcal{F}} , \quad (28)$$

while in the ingoing case we have

$$T = a\tau - \int dR \frac{\sqrt{1-\mathcal{F}a^2}}{\mathcal{F}} . \quad (29)$$

For the concrete calculation we shall write the full states as a product of single-shell states where the radial variable r is assumed to consist of discrete points separated by a distance σ . (The continuum limit is obtained for $\sigma \rightarrow 0$.) As in [5], the Bogolyubov coefficient β is calculated for each shell separately. In the discrete case, we replace Γ by the dimensionless variable 2ω and indicate the dependence on ω by an index. (The factor 2 is motivated by the fact that $\Gamma = 2GM'$.) We omit the shell index and write the corresponding wave functions as $\psi_\omega(T, R)$. We then define β to read

$$\beta_\omega \equiv \int_F^\infty dR \sqrt{g_{RR}} \Psi_{e\omega}^* \Psi_{c\omega}^+ , \quad (30)$$

where g_{RR} is the RR -component of the DeWitt metric, as it can be read off (5) where the inverse of the DeWitt metric is l_P^{-4} times the prefactor of the term $\delta^2 \Psi / \delta R^2$. We thus have $g_{RR} = \mathcal{F}^{-1}$; performing then the required coordinate transformation from the variables (τ, R) to (T, R) gives the result $\sqrt{g_{RR}} = (a\mathcal{F})^{-1}$ which has to be used in the calculation of the Bogolyubov coefficient β .

Inserting now Ψ_{ω}^{-*} and Ψ_{ω}^{+} into (30), we get

$$\beta_{\omega} = \int_F^{\infty} dR (a\mathcal{F})^{-1} \exp\left(-\frac{2i\omega\sigma}{l_P^2} [T + \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}}] - 2i\omega\sigma \frac{C}{a} \int^R dR \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}} + 2\omega\sigma\delta(0)\tilde{H}\right) \quad (31)$$

We note that this expression is independent of \tilde{G} .

In the following, we shall employ a “DeWitt regularization” and set $\delta(0) = 0$. Whether this can consistently be done at the most fundamental level is, however, not clear at this stage; here, it is merely used as a formal recipe. Recalling that $(a\mathcal{F})^{-1} = R/a(R-F)$, we then get

$$\beta_{\omega} = a^{-1} \exp\left(-\frac{2i\omega\sigma T}{l_P^2}\right) \int_F^{\infty} dR \frac{R}{R-F} \times \exp\left(-\frac{2i\omega\sigma}{l_P^2} \left[1 + l_P^2 \frac{C}{a}\right] \frac{\sqrt{1-a^2\mathcal{F}}}{\mathcal{F}}\right). \quad (32)$$

As in [5], we introduce the dimensionless integration variable

$$s = \sqrt{\frac{R}{F}} - 1$$

and get

$$\beta_{\omega} = 2Fa^{-1} \exp\left(-\frac{2i\omega\sigma T}{l_P^2}\right) \int_0^{\infty} ds \frac{(1+s)^3}{s^2+2s} \times \exp\left(-\frac{4i\omega\sigma F}{l_P^2} \left[1 + l_P^2 \frac{C}{a}\right] \int^s ds \frac{(s+1)^2}{s^2+2s} \sqrt{(1+s)^2 - a^2(s^2+2s)}\right). \quad (33)$$

Up to higher orders of the Planck length squared in the exponent, this is so far an exact expression. As in [5], we now assume that the s -integral from zero to infinity is dominated by its contribution near $s = 0$, that is, near the horizon; this is also the assumption in the standard derivation of the Hawking effect [17]. Using therefore in (33) the approximation

$$\frac{(s+1)^2}{s^2+2s} \sqrt{(1+s)^2 - a^2(s^2+2s)} = \frac{1}{2s} \left(1 + \left[\frac{5}{2} - a^2\right] s + \mathcal{O}(s^2)\right), \quad (34)$$

we get

$$\beta_{\omega} \approx Fa^{-1} \exp\left(-\frac{2i\omega\sigma T}{l_P^2}\right) \int_0^{\infty} ds s^{-1-\frac{2i\omega\sigma F}{l_P^2}(1+l_P^2 \frac{C}{a})} \times \exp\left(-\frac{2i\omega\sigma F}{l_P^2} \left[1 + l_P^2 \frac{C}{a}\right] \left[\frac{5}{2} - a^2\right] s\right). \quad (35)$$

To evaluate this integral, we use the formula [18]

$$\int_0^{\infty} dx x^{\nu-1} e^{-(p+iq)x} = \Gamma(\nu)(p^2+q^2)^{-\nu/2} e^{-i\nu\arctan(q/p)},$$

which is, in particular, applicable to the case $p = 0$ and $0 < \text{Re } \nu < 1$. (We insert a small positive value for $\text{Re } \nu$, which we let go to zero after the integration.) Using, moreover,

$$\Gamma(-iu)\Gamma(iu) = \frac{\pi}{u \sinh \pi u}$$

(with real u), we get

$$|\beta_{\omega}|^2 \approx \frac{2\pi F^2}{a^2 y} \frac{1}{e^{2\pi y} - 1} \quad (36)$$

with

$$y = \frac{2\omega\sigma F}{l_P^2} \left(1 + l_P^2 \frac{C}{a}\right). \quad (37)$$

Substituting $\sigma\omega$ by $G\Delta\epsilon$,¹ where $\Delta\epsilon$ is the energy of a shell, and introducing the physical frequency $\Omega = \Delta\epsilon/\hbar$, we arrive at the final result

$$|\beta_{\Omega}|^2 = \frac{2\pi GM}{\Omega a^2 (1 + l_P^2 \frac{C}{a})} \frac{1}{\exp\left(\frac{\hbar\Omega}{k_B T_H}\right) - 1} \quad (38)$$

with the quantum-gravity corrected Hawking temperature

$$k_B T_H = \frac{\hbar}{8\pi GM (1 + l_P^2 \frac{C}{a})}. \quad (39)$$

A number of remarks are in order. First, the meaning of the prefactor in the expression for $|\beta_{\Omega}|^2$ (which depends mildly on Ω) is unclear. It is certainly connected with the greybody factors, but without a clear-cut normalization of the quantum states, its interpretation remains incomplete. Secondly, different from the earlier papers, we have calculated the overlap of the quantum states in (30) for coinciding frequencies ω only. The reason is that an (approximate) thermal spectrum only occurs in that case. Unlike the highest order l_P^0 , taking here also into account two frequencies ω and ω' corresponding to two different shells, the integration over ω' would spoil the thermality. The results (38) and (39) thus remain valid only as far as the interaction between different shells is subdominant. Thirdly, when taking the next order in the Planck-mass expansion into account, we expect that the term $1 + l_P^2 C/a$ in the denominator of (39) is augmented by a term proportional to $l_P^4 D/a$, where D occurs in (22).

¹ Recall that ω is the discretized version of $\Gamma/2 = GM'$, so $\sigma\omega$ corresponds to $G\Delta M \equiv G\Delta\epsilon$.

The form of the temperature given in (39) for $a = 1$ was obtained earlier for the case of the Schwarzschild black hole in [10]. It was calculated there by using the quantum tunneling method beyond the semiclassical approximation.

We emphasize that in the previous papers [5, 7] no such quantum gravitational correction to the Hawking temperature has been found, since calculations have led to an exact solution with semiclassical form.

Substituting now (17) in (19) we obtain the expression for S_1 as,

$$S_1(R, \tau) = -\frac{\alpha_1}{(GM)^2} \tau + a \frac{\alpha_1}{(GM)^2} \left[\frac{\sqrt{R([1-a^2]R + a^2F)}}{1-a^2} - \frac{a^2F}{(1-a^2)^{3/2}} \ln \left(\sqrt{(1-a^2)R} + \sqrt{(1-a^2)R + a^2F} \right) \right] - i \frac{\delta(0)}{2\Gamma} (2 \ln(R-F) - \ln(R + a^2[F-R]) - \ln R) \quad (40)$$

We discuss in the following a method developed in [11] to find the value of the dimensionless constant α_1 . Consider for that purpose a constant scale transformation of the coefficients of the metric (1), given by (cf. also [19])

$$\bar{g}_{\mu\nu} = k g_{\mu\nu} . \quad (41)$$

Under this transformation we have from (1),

$$\bar{R} = k^{\frac{1}{2}} R . \quad (42)$$

For the Einstein equations to remain invariant under this scale transformation, F should according to (3) transform as

$$\bar{F} = k^{\frac{1}{2}} F , \quad (43)$$

and τ should transform as

$$\bar{\tau} = k^{\frac{1}{2}} \tau . \quad (44)$$

Therefore, (4) yields

$$\bar{\mathcal{F}} = \mathcal{F} , \quad (45)$$

and since $F = 2GM$, GM transforms as

$$\overline{(GM)} = k^{\frac{1}{2}} (GM) . \quad (46)$$

For the Wheeler–DeWitt equation (5) to be invariant under this scale transformation, we must have in addition the following transformations:

$$\bar{\Psi} = k \Psi , \quad (47)$$

$$\bar{\Gamma} = k^{-\frac{1}{2}} \Gamma , \quad (48)$$

where (42) and (45) have been used.

Now from the expression for Ψ , Eq. (7), we have

$$\frac{\hbar}{i} \frac{\delta \Psi(\Gamma, R, \tau)}{\delta X^\mu(r)} = \frac{\Gamma(r)}{2G} \frac{\partial S(R, \tau)}{\partial X^\mu(r)} \Psi(\Gamma, R, \tau) , \quad (49)$$

where $\mu = 0, 1$ with $X^0 = \tau$ and $X^1 = R$. Because we have in (49) a functional derivative on the left-hand side and an ordinary derivative multiplied by $\Gamma(r)$ on the right-hand side, S is not the action. In fact, as we have seen, it is the quantity (11) which has the correct physical dimension mass times length and which is equal to the action of this model. The first-order action is therefore given by

$$\mathcal{A}_1 = \frac{l_P^2}{2G} \int dr \Gamma S_1(R, \tau) . \quad (50)$$

We have seen that the relevant part for the recovery of Hawking radiation was the first, purely τ -dependent, term in (40). Because this part does not contain the gravitational variables, it can for our purpose be considered as the matter (dust) action which we shall call \mathcal{A}_1^m . We thus have

$$\mathcal{A}_1^m = -\frac{l_P^2}{2G} \int dr \Gamma \frac{\alpha_1 \tau}{(GM)^2} . \quad (51)$$

Hence, under the transformations (44), (46), and (48), \mathcal{A}_1^m transforms as

$$\bar{\mathcal{A}}_1^m = -\frac{l_P^2}{2G} \int dr \bar{\Gamma} \frac{\alpha_1 \bar{\tau}}{(GM)^2} = k^{-1} \mathcal{A}_1^m \simeq (1 - \delta k) \mathcal{A}_1^m , \quad (52)$$

where we have assumed that α_1 and r do not scale, and $k \simeq 1 + \delta k$. Hence,

$$\delta \mathcal{A}_1^m = \bar{\mathcal{A}}_1^m - \mathcal{A}_1^m = -\mathcal{A}_1^m \delta k . \quad (53)$$

Now using the definition of the energy–momentum tensor,

$$\begin{aligned} \int d^4x \sqrt{-g} T_\mu^\mu &= \frac{2\delta \mathcal{A}_1^m}{\delta k} = -2\mathcal{A}_1^m \\ &= \hbar \int dr \Gamma \frac{\alpha_1 \tau}{(GM)^2} \longrightarrow 2\omega \sigma \frac{\hbar \alpha_1 \tau}{(GM)^2} , \end{aligned} \quad (54)$$

where in the last step, we pass to the discretized version by replacing Γ by the dimensionless variable 2ω as before.

Therefore, by a simple interposition of (54), we obtain,

$$\alpha_1 = \frac{(GM)^2}{2\hbar \omega \tau \sigma} \int d^4x \sqrt{-g} T_\mu^\mu . \quad (55)$$

Next, our task is to find the value of τ . For the *contracting cloud* case, τ is given by the relation (29), which can be written in a rearranged form:

$$\tau = \frac{T}{a} + \frac{1}{a} \int dR \frac{\sqrt{1 - \mathcal{F} a^2}}{\mathcal{F}} . \quad (56)$$

Now in the calculation of $\left|\beta_\omega\right|^2$ (see equations (35) and (36)), which eventually gives the flux for the model, the first term in (56) is inconsequential, since it occurs as a phase factor ($\sim e^{i\alpha T}$) which yields unity on taking the modulus. On the other hand, as we have seen, the terms which contribute to the flux come from the integration term. Therefore, for the present calculation, considering only the last term of (56) and using the fact that $\mathcal{F} = 1 - \frac{F}{R}$ and then substituting $s = \sqrt{\frac{R}{F}} - 1$, we obtain,

$$\tau = 2F \int ds \frac{(s+1)^2}{s^2+2s} \sqrt{(s+1)^2 - a^2(s^2+2s)}. \quad (57)$$

If this integral is supposed to run from 0 to ∞ (since R runs from F to ∞), it is certainly divergent. In order to make contact with the expressions in earlier papers, we make the following heuristic considerations, which can be viewed as a regularization prescription. Restricting attention to the relevant regime near $s = 0$, use of the approximation (34) in the above yields,

$$\tau \simeq 2F \int ds \frac{1}{2s} \left[1 + \left(\frac{5}{2} - a^2 \right) s \right] = F \ln s + \mathcal{O}(s). \quad (58)$$

Interpreting s as a complex variable and recalling $\ln s = \ln |s| + i \arg s$, we can define a “Euclidean time” τ_E by

$$\tau_E = i\tau = -\pi F = -2\pi(GM). \quad (59)$$

Finally, substituting this value for τ_E in the Euclideanized version of (55), we obtain our cherished expression,

$$\alpha_1 = -\frac{GM}{4\pi\hbar\omega\sigma} \int d^4x_E \sqrt{-g} T_\mu^\mu. \quad (60)$$

This shows that α_1 is related to the “one-loop trace anomaly” of the energy-momentum tensor. A similar result was obtained earlier in [8, 9, 11–13] for the eternal black hole case.

To conclude, we mention that a quantum gravitational correction to the Hawking temperature from the LTB model was established through a semiclassical approximation scheme employed in [10]. Here, no special factor ordering [4, 5] was chosen, which was a crucial step to

obtain such a correction. Instead, we considered all the terms in the expansion for S (9). This led to several equations corresponding to different orders in l_P^2 . In this paper, only terms upto $\mathcal{O}(l_P^2)$ were considered. The equations were solved by a special ansatz. After getting the solutions for the states upto $\mathcal{O}(l_P^2)$, the “De Witt regularization” was employed. This regularization enforced $\delta(0) = 0$. The explicit calculation of the Bogolyubov coefficient near the horizon led to the emission spectrum. The corrected Hawking temperature was then automatically identified. It contained an unknown variable “ C ”. Dimensional arguments then helped us to fix “ C ”, apart from a dimensionless constant.

The last part of the paper was actually devoted to fix the dimensionless constant appearing in “ C ”. It was done by a constant scale transformation of the metric coefficients (1). A detailed analysis showed that it was related to the one loop trace anomaly of the energy-momentum tensor for the dust (matter).

It must be emphasized that a similar result was obtained by Hawking [19] for an eternal black hole space-time by taking into account the one loop correction to the partition function due to the fluctuations of the scalar fields on the black hole space-time. Exactly the same result was also derived later on by different methods [8, 9, 11–13]. Here our analysis was done in the spirit of the quantum tunneling method employing the WKB approximation [10–13]. Indeed the special ansatz (16) used here closely resembles the Hamilton-Jacobi splitting of the one particle action $S(t, r) = \omega t + \tilde{S}(r)$. Such a similarity of our result with Hawking’s finding [19] may be due to the equivalence of the path integral with the WKB ansatz upto $\mathcal{O}(l_P^2)$, a result that has been established earlier in quite general terms [20]. That this connection also holds in the black hole context is a new observation.

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